

CONSTRUCTION OF COSTAS SEQUENCES AND COSTAS ARRAYS

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Introduction

Costas sequences are the best sequences for applications in frequency-hopping radar and sonar systems. As a frequency-hop pattern for radar or sonar, a Costas array has an optimum ambiguity function, since any translation of the array parallel to the coordinate axes produces at most one out-of-phase coincidence.

A Costas sequence $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ is a sequence which is a permutation of integers $1, 2, 3, \dots, n$ satisfying the property $\alpha_{s+k} - \alpha_s \neq \alpha_{t+k} - \alpha_t$ for every s, t and k , such that $0 \leq s < t < t + k \leq (n - 1)$. An array that results from a Costas sequence in this way is called a Costas array (Golomb, 1984). A Costas array of order n is an $n \times n$ array of dots and blanks which satisfies:

- (i) There are n dots and $n(n - 1)$ blanks, with exactly one dot in each row and column.
- (ii) All the segments between pairs of dots differ in length or in slope.

Let $C(n)$ denote the number of distinct $n \times n$ Costas arrays, $c(n)$ denote the number of $n \times n$ Costas arrays, inequivalent under

the dihedral group of rotations and the reflections of the square and $s(n)$ denotes the number of $n \times n$ Costas arrays which are symmetric across a diagonal and inequivalent under the dihedral group of rotations

and reflections of the square. For example a Costas array of order 4 is shown in Fig 1.

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		•	
			•
	•		

$C(4)=12, c(4)=2$
and $s(4)=1$.

Fig 1. A Costas array of order 4

It is conjectured that $n \times n$ Costas arrays exists for every positive integer n (Golomb and Taylor, 1984). There are several methods of construction of Costas arrays. They include Welch construction, the Lempel construction and the Golomb construction. In this research paper we have focused on the Welch construction.

In addition, we have considered about constructing Costas like sequences using the set of non-zero quadratic residues of a finite field with p elements ($GF(p)$). $a \in GF(p)$ is said to be a quadratic residue modulo p if $(a, p) = 1$ and $x^2 \equiv a \pmod{p}$ has a solution. It is well known that quadratic residues form a difference set. For example if $GF(p) = Z_7$, $1, 2, 4$ are quadratic residues mod 7 and $\{1, 2, 4\}$ is a difference set.

Methodology

Let us consider a finite field with q elements, denoted by $GF(q)$ where q is a prime power. The element α in $GF(q)$ is called primitive if it satisfies:

- (i) For every non-zero element x in $GF(q)$ there is an integer i such that $\alpha^i = x$.
- (ii) $\alpha^i = \alpha^k$ in $GF(q)$ if and only if $i \equiv k \pmod{q-1}$.

Welch construction

Let p be a prime and α a primitive element in $GF(p)$. Then $\alpha^1, \alpha^2, \dots, \alpha^{p-1} = 1$ is a Costas sequence. Moreover, any circular shift of this sequence is also a Costas sequence. For an example $\alpha^i, \alpha^{i+1}, \alpha^{i+2}, \dots, \alpha^{i+p-2}$ is also a Costas sequence. This construction is called an exponential Welch construction and the inverse permutation is called a log Welch construction (Moreno and Golomb, 1996).

Consider $GF(11) = \{0,1,2,3,\dots,10\}$. 2 is a primitive element since $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 5, 2^5 = 10, 2^6 = 9, 2^7 = 7, 2^8 = 3, 2^9 = 6, 2^{10} = 1 \pmod{11}$.

The sequence $\{2,4,8,5,10,9,7,3,6,1\}$ is a Costas sequence since it satisfies the property of a Costas sequence and the resulting array is a Costas array (Fig 2).

Results

Any circular shift of the sequence $\{2,4,8,5,10,9,7,3,6,1\}$ is also a Costas sequence. So,

$$\{2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^8, 2^9, 2^{10}\} = \{4, 8, 5, 10, 9, 7, 3, 6, 1, 2\}$$

is a Costas sequence and it results a Costas array (Fig3).

	1	2	3	4	5	6	7	8	9	10
1										■
2	■									
3								■		
4		■								
5				■						
6									■	
7								■		
8			■							
9						■				
10					■					

Fig 2. The Costas array corresponding to the Costas sequence $\{2,4,8,5,10,9,7,3,6,1\}$.

Fig 3. The Costas array corresponding to the Costas sequence {4,8,5,10,9,7,3,6,1,2}.

	1	2	3	4	5	6	7	8	9	10
1									■	
2										■
3							■			
4	■									
5			■							
6								■		
7						■				
8		■								
9					■					
10				■						

Table 1. Some C(n) values (n prime)

N	C(n)
1	1
3	4
5	40
7	200
11	4368

C(n) values for some Costas arrays of order n, when n is a prime is given in Table 1.

Also we can show that the set of non-zero quadratic residues D of GF(p) has a nice property. That is, we can always find a generating element α such that the resulting sequence satisfies the Welch construction condition. Since the length of the sequence

which we constructed has length less than the alphabet, the corresponding array is not a square (n × n) array. So, we can construct Costas like sequences.

Example 1: In GF(7), D= {1,2,4} with α = 2, the sequence {2,4,1} is a Costas like sequence. By adding quadratic non-residues a Costas sequence {2,4,1,6,5,3} can be constructed.

Example 2: In GF(11), D={1,3,4,5,9} with α = 3, the sequence {3,9,5,4,1} is a Costas like sequence. By adding quadratic non-residues sequence {3,9,5,4,1,8,2,6,7,10} can be constructed.

Conclusions

Costas sequences can be constructed for GF(p), where p is a prime using Welch construction. Costas like sequences can be constructed for GF(p) and extend to Costas sequences. Further this result can be generalized for GF(q), where q is a prime power.

References

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