THE TIME-PATH OF THE PRICE FUNCTION IN A DYNAMIC MARKET-MODEL

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Introduction

Due to socio-economic changes, environmental hazards and political catastrophe, the time-path of the price function $P(t)$ in a market-model fluctuates rapidly and our aim is to predict the long-term behaviour of the price-function (Convergence and Stability) via dynamic modelling. In this study we mathematize the mechanics of such scenarios by incorporating the random effects with the use of Brownian-Motion modeled by a generalized-function B_t. In this paper we shall investigate *the time path of the price function* in the light of Ito-calculus and we wish to predict the asymptotic-nature of the Intertemporal Equilibrium Price (IEP) in the market-model.

Model Building

A micro dynamic market-model is established in the continuous-time context based on the following assumptions which are motivated by usual static micro market-models (Chiang, 1984):

- The usual linear demand and supply functions in a static micro market-model are modified by incorporating the price-trends and random effects via the terms $\frac{dP}{dt}$ and $\frac{dB_t}{dt}$ with the use of appropriate multipliers σ_j and μ_j $(j=1,2)$.
- The random disturbances are included in the form of white noise $W(t)$ which is directly proportional to the Ito-derivative $\frac{dB_t}{dt}$ of the generalized function B_t (Michael, 2001).
- The rate of change of price at any moment is directly proportional to the excess demand function $E_{dt}\ (=Q_d - Q_s\ )$.

Next, we shall mathematize the above assumptions to obtain the governing differential equation of the model:

$$Q_d = \alpha - \beta\, P(t) + \sigma_1 \frac{dP}{dt} + \mu_1 \frac{dB_t}{dt} \quad \text{,where } (\alpha > 0, \beta > 0) \qquad (1)$$

$$Q_s = -\gamma + \delta\, P(t) + \sigma_2 \frac{dP}{dt} + \mu_2 \frac{dB_t}{dt} \quad \text{,where } (\gamma > 0, \delta > 0) \qquad (2)$$

$$\frac{dP}{dt} = k(Q_d - Q_s\ )\ ;(\kappa > 0) \qquad (3)$$

Here $\sigma_1, \mu_1; \sigma_2, \mu_2$ are real parameters to be determined in the due course and they embody the new trends of the building process.

Using the equations (1), (2) and (3), we get the governing equation of the model in the form,

$$(1 - \kappa\sigma_1 + \kappa\sigma_2)\frac{dP}{dt} = \kappa\ [(\alpha+\gamma) - (\beta+\delta)P(t)] + \kappa(\mu_1 - \mu_2)\frac{dB_t}{dt}\ .$$

Here $P(0)=P_0>0$ almost surely; and we need to impose the condition $1>\kappa(\sigma_1-\sigma_2)$ for the future purposes. Hence the governing equation of the model can be written in the differential form

$dP=-APdt+Cdt+HdB_t$, where

$$A=\frac{k(\beta+\delta)}{1-k\sigma_1+k\sigma_2},\quad C=\frac{k(\alpha+\gamma)}{1-k\sigma_1+k\sigma_2}$$

are positive parameters and

$$H=\frac{k(\mu_1-\mu_2)}{1-k\sigma_1+k\sigma_2} \tag{4}$$

Model Usage

Now we shall indicate the key steps of the process of obtaining the time-path of the price function *P(t)*. For this purpose we write (4) in the form

$$\exp(At)\,dP+\exp(At)\,APdt=\exp(At)[Cdt+HdB_t] \tag{5}$$

To convert the left hand side of (5)to the exact differential form $d(\exp(At)P)$, we use the Ito formula for the function. $g:[0,\infty)\times\Re\to\Re$ given by $g(t,x)=\exp(At)x$.

Hence, we obtain

$$d(\exp(At)P)=A\exp(At)Pdt+\exp(At)\,dP$$

Using (3.1), we get

$$d(\exp(At)P)=\exp(At)Cdt+\exp(At)\,HdB_t$$

$$\Rightarrow\int_0^t d(\exp(At)P)=\int_0^t\exp(As)Cds+\int_0^t\exp(As)\,HdB_s$$

$$\Rightarrow\exp(At)P-P_0=C\left[\frac{\exp(As)}{A}\right]_0^t+\int_0^t\exp(As)\,HdB_s$$

$$\Rightarrow P(t)=\exp(-At)\left[P_0+\frac{C}{A}(\exp(At)-1)+\int_0^t\exp(As)\,HdB_s\right] \tag{6}$$

Utilizing the method of integration by parts in the Ito-integral (Michael, 2001),

$\int_0^t\exp(As)\,HdB_s$, *we get*

$$\int_0^t\exp(As)\,HdB_s=\exp(At)\,H\,B_t-\int_0^t A\exp(As)\,HB_sds$$

Substituting in (6), we arrive at

$$P(t)=\left(\frac{C}{A}+HB_t\right)+\exp(-At)\left[P_0-\frac{C}{A}-\int_0^t A\exp(As)\,HB_sds\right]$$

Thus the time-path of the price function takes the explicit form,

$$P(t)=\left(\frac{\alpha+\gamma}{\beta+\delta}\right)+\frac{k(\mu_1-\mu_2)}{1-k\sigma_1+k\sigma_2}B_t+e^{\frac{-k(\beta+\delta)t}{1-k\sigma_1+k\sigma_2}}\left[P_0-\left(\frac{\alpha+\gamma}{\beta+\delta}\right)-\int_0^t\frac{k^2(\beta+\delta)(\mu_1-\mu_2)}{(1-k\sigma_1+k\sigma_2)^2}e^{\frac{k(\beta+\delta)s}{1-k\sigma_1+k\sigma_2}}B_sds\right]$$

Results and Discussion

$$E\left[\int\exp(As)HdB_s\right]=0$$ and by (6) we have,

$$P(t)=\frac{C}{A}+\exp(-At)\left[P_0-\frac{C}{A}+\int_0^t\exp(As)\,HdB_s\right]$$

Thus the expected value of the time-path of the price function is given by

$$E(P(t))=\frac{C}{A}+E(P_0)e^{-At}-\frac{C}{A}e^{-At};$$

$$\text{ie } E(P(t))=\left(\frac{\alpha+\gamma}{\beta+\delta}\right)+E(P_0)e^{\frac{-k(\beta+\delta)t}{1-k\sigma_1+k\sigma_2}}-\left(\frac{\alpha+\gamma}{\beta+\delta}\right)e^{\frac{-k(\beta+\delta)t}{1-k\sigma_1+k\sigma_2}} \tag{7}$$

Note that $1-k\sigma_1+k\sigma_2>0$ and hence

$$E(P(t))\to\left(\frac{\alpha+\gamma}{\beta+\delta}\right)\quad\text{as } t\to\infty;$$

providing a stochastically stationary value for the price function in the long-run.

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$$V(P(t))=V\left(\frac{C}{A}\right)+V\left(P_0 e^{-At}\right)-V\left(e^{-At}\frac{C}{A}\right)+ V\left(e^{-At}\int_0^t e^{As}\,HdB_s\right)$$

$$=e^{-2At}\left\{V(P_0)+E\left(\int_0^t e^{As}\,HdB_s\right)^2-\left[E\left(\int_0^t e^{As}\,HdB_s\right)\right]^2\right\}$$

$$=e^{-2At}\left\{V(P_0)+E\left(\int_0^t e^{As}\,HdB_s\right)^2\right\}$$

Moreover,

$$E\left(\int_0^t e^{As}\,HdB_s\right)^2=E\left(\int_0^t e^{2As}\,H^2ds\right)$$

and we have,

$$V(P(t))=e^{-2At}\left\{V(P_0)+E\left(\int_0^t e^{2As}\,H^2ds\right)\right\}$$

The use of the Holder's inequality provides an upper bound for the variance; namely, using the inequality

$$V(P(t))\le e^{-2At}\left\{V(P_0)+E\left[\left(\int_0^t e^{4As}ds\right)^{1/2}\left(\int_0^t H^4ds\right)^{1/2}\right]\right\}$$

we arrive at

$$V(P(t))\le V(P_0)+\left\|H^2\right\|_2\frac{1}{\sqrt{2A}}\ .$$

Conclusions

(i) $P(t)=\left(\dfrac{\alpha+\gamma}{\beta+\delta}\right)+e^{\frac{-k(\beta+\delta)t}{1-k\sigma_1+k\sigma_2}}\left[P_0-\left(\dfrac{\alpha+\gamma}{\beta+\delta}\right)\right]$

under the hypothesis $\mu_1=\mu_2$.

(ii) $E(P(t))$ converges to the stationary point $\left(\dfrac{\alpha+\gamma}{\beta+\gamma}\right)$, provided $\sigma_1-\sigma_2<\dfrac{1}{\kappa}$.

(iii) $\{V(P(t))-V(P_0)\}$ is bounded for all $t>0$.

Thus the IEP is stochastically-stationary and the market model is asymptotically-stable.

Finally we wish to mention that, once the data collection is completed via the field work the practical aspects of the theory can be tested and the policy implications will be proposed accordingly.

References

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