A STUDY OF FUNCTIONS OF BOUNDED FLUCTUATION AND EXTENSION TO METRIC SPACES

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Introduction

The class of functions of bounded variation is a class which plays a major role in analysis, especially in theory of integration and in functional analysis. In this study, the difficulty of making an analogous definition of bounded variation in higher dimension is discussed. The class of functions of bounded fluctuation is a class of functions having similar properties. The equivalence of the two definitions is established in the case of one real variable, and the notion of bounded fluctuation is analogously extended to metric spaces. Finally, a special feature of functions of bounded fluctuation in metric spaces is studied in comparison with functions of a real variable.

Functions of Bounded Variation

Definition: Let f be a real valued function defined on a compact interval $[a,b]$ and let $\wp$ denote the collection of all partitions of $[a,b]$. If there exists a constant M such that $\sum_{i=0}^{n-1} | f(x_i) - f(x_{i-1}) | \leq M$ for all partitions $P = \{x_0, x_1, ..., x_n\}$ in $\wp$, then f is said to be of bounded variation on $[a,b]$, and the total variation $V_f[a,b]$ of f on $[a,b]$ is defined as the function $V_f[a,b] = \sup_{P \in \wp} \sum_{i=1}^{n-1} | f(x_i) - f(x_{i-1}) |$. The class of functions of bounded variation on the interval $[a,b]$ is denoted by *BV[a,b]*.

Properties of Functions of Bounded Variation: A significant feature of the class of BV is its connection to monotone functions. In 1904, H. Lebesgue established the differentiability of monotone functions in his collection *Sur l'intégration et la recherché des fonctions primitives*, and as a final result, he extended this to a larger class of functions, the class of BV. The following theorem allows us to express a function of bounded variation as the difference of two monotonically increasing functions:

Theorem 1: Jordan Decomposition (Folland, 1999). A function f is of bounded variation if and only if it is expressible as a difference of two monotonically increasing functions on $[a,b]$.

The Lebesgue differentiation theorem for monotone functions is as follows:

Theorem 2: Lebesgue Differentiation Theorem (Riesz and Nagy, 1954). Let $f:[a,b] \rightarrow \Re$ be a monotonically increasing function. Then f is differentiable except at most on a set of measure zero.

Following is a theorem concerning continuity of BV functions.

Theorem 3: (Folland,1999). Let $f:[a,b]\to\Re$ be a function of bounded variation on $[a,b]$, then f is continuous except at most on a countable set, also one sided limits of f exist everywhere.

The class of functions of bounded variation is important in analysis of Fourier series. The following theorem, (proved by C. Jordan in 1884) establishes the necessity of being BV for a function to uniformly converge to its Fourier series:

Theorem 4: Dirichlet-Jordan theorem (Pierce and Velleman, 2006). Let $f:[a,b]\to\Re$ be a periodic function that is of bounded variation on $[-\pi,\pi]$. Then at every point x, the Fourier series of f at x converges to $1/2(f(x^+)+f(x^-))$; in particular, when f is continuous at x, Fourier series of f at x sums to $f(x)$. Also If f is continuous at every point of a closed interval I, then the Fourier series converges uniformly.

Functions of Bounded Fluctuation

The concept of bounded fluctuation (Clarkson and Adams, 1933) is a concept very similar to that of bounded variation. Here we take the maximum fluctuation (oscillation) of a function in a closed subinterval. The two concepts can be easily proved to be equivalent in the case of a real valued function of one real variable (Clarkson and Adams, 1933).

Definition: let $f:[a;b]\subseteq\mathbb{R}$ be a function, and let $I=[c,d]\subseteq[a,b]$. the oscillation of f in $[c,d]$ is defined as the value $\sup_{x,y\in I}|f(x)-f(y)|$. This is denoted by $O_f(I)$. A function $f:[a;b]\subseteq\mathbb{R}$ is said to be of bounded fluctuation on [a; b] if there exist a constant M such that $\sum O_f(I_n)\leq M$ whenever $\{I_n\}$ is a finite collection of non-overlapping closed intervals of $[a,b]$. The supremum of the fluctuation is defined as $Fl_f[a,b]=\sup_{\{I_n\in\wp\}}\{O_f(I_n)|\{I_n\}$ is a finite collection of non-overlapping closed intervals of $[a,b]\}$

The fluctuation function of $f:[a,b]\to\Re$ is the function defined by $Fl_f(x)=Fl_f[a,x]$.

In several ways, the notion of bounded fluctuation is similar to the notion of bounded variation. A contrasting difference is that, the notion of bounded variation is dependent on the order properties of the domain $[a,b]$, but bounded fluctuation does not depend on the order properties of the domain. Instead, it entirely depends on the distance-like (metric) properties of the range of the function. The interesting fact is that, though bounded variation is difficult to analogously extend to the case of several variables and beyond, the notion of bounded fluctuation is not. In fact, very slight modification of the definition of bounded fluctuation allows us to define the same concept from functions from compact metric spaces to metric spaces. The definition of bounded fluctuation from a compact metric space to a metric space can be formulated as follows:

Definition: Let (X,d) be a compact metric space and (Y,ρ) be a metric space. let $f: D \to Y$ be a function, where D is a closed subset of X. Define the oscillation of f in $I \subseteq D$, where I is closed in D by $O_f(I) = \sup_{x,y\in I}\{\rho(f(x), f(y))\}$. The function f is said to be of bounded fluctuation on D if there exists $M \in \mathbb{R}$ such that $\sum O_f(I_n) \leq M$ for all finite collections I_n of non-overlapping (with disjoint interiors) closed subsets of D.

The supremum of $O_f(I_n)$ is called the fluctuation of f in D and is denoted by $Fl_f(D)$. The fluctuation function of a function $f : X \to Y$, where X and Y are metric spaces is defined as follows, let $f: D \to Y$ be a function, where D is a closed subset of X. Consider the closed ball $B_d(a;r)$, where r is a positive real number. Let, $B_x = B_d(a;\bar{x}) \cap D \subseteq D$ where $\bar{x} = d(a,x)$. For a function f of bounded fluctuation, the fluctuation function is $F(x) = Fl_f(B_x)$. That is, the supremum of the fluctuation in B_x. Now we show that the fluctuation function defined above is a monotone function. Let the point x_0 be in the boundary of D as the initial point. Then $F(x_0) = 0$ and whenever $B(x_1) \subseteq B(x_2), d(x_0,x_1) \leq d(x_0,x_1)$. Thus $Fl_f(B(x_1)) \leq Fl_f(B(x_2))$, therefore the function increases as we move away from the point x_0. According to the definition of monotone functions (Price, 1940), the function F is monotone.

Discussion

The notion of bounded variation for real valued functions cannot be extended analogously to higher dimensions, the higher dimensional bounded variation is defined using the directional derivative of the vector Radon measure. The notion of bounded fluctuation is equivalent to bounded variation for real valued functions. The notion of bounded fluctuation, being independent of order properties of the domain, can be extended to metric spaces. A fluctuation function can be defined for functions from a metric space to another, and this function is a monotone function. Further research should be implemented to study continuity and differentiability properties of functions of bounded fluctuation in metric spaces.

References

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