GROUP REPRSENTATIONS AND CHARACTERS

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Introduction

In mathematics, a representation of a group G describes a way of visualizing G as a group of matrices. To be precise, a representation is a homomorphism from G into a general linear group $GL(n,C)$, which is the group of invertible $n \times n$ matrices with entries in C (the set of complex numbers) . A character is obtained from a representation by taking the trace of each matrix. The function $\chi : G \to C$ is called the character of the representation. When G has a k conjugacy classes, then G has k distinct irreducible representations over C. Hence, the values of the characters can be written as an array, known as a character table. Typically, the rows are given by the irreducible representations and the columns are given the conjugacy classes (Gordo and Martin, 1993).

Moreover, an important consequence of these mathematical relationships is that each point group can be decomposed into symmetry patterns known as irreducible representations which help in analyzing many molecular properties (Gray and Donald, 1991).

The character table of a Finite Abelian group can be constructed by using a known formula. We have shown that the character table of any finite direct product of cyclic group of order 2 into itself can be obtained by Sylvester's construction of a Hadamard matrix.

Methodology

Let G be a cyclic group of order n $(C_n = \langle a : a^n = 1 \rangle)$. In this case each element forms a conjugacy class by itself. Then the n irreducible representations of G over C are $\chi^{(r)}$, $(0 \le r \le n-1)$,here

$\chi^{(r)}(a^s) = \exp(2\pi irs / n), (0 \le s \le n-1)$.

The irreducible representations of direct products $C_{n_1} \times C_{n_2} \times \ldots \times C_{n_m}$ where $n_1, n_2, \ldots, n_m \in \mathbf{Z}^+$ are given by

$$\chi^{(r)}(x) = \exp\left(2\pi i \sum_{\mu=1}^{m} \frac{b_\mu r_\mu}{n_\mu} \right).$$

Here $x = a_1^{b_1} a_2^{b_2} \ldots a_m^{b_m}$ where $a_\mu \in C_{n_\mu}$ and $r = [r_1, r_2, \ldots, r_m]$ where $0 \le r_\mu, b_\mu < n_\mu$. Since every Finite Abelian group is isomorphic to a direct product of cyclic groups, this covers the irreducible representations of all Finite Abelian groups. When N is a normal subgroup of group G and $\sigma_0(Nx)$ where $x \in G$ is a representation of the factor group G/N of degree m, then the representation σ of G is defined as $\sigma(x) = \sigma_0(Nx)$ which is '*lifted*' from G/N. If $\varphi_0(Nx)$ is the character of $\sigma_0(Nx)$, then $\varphi(x) = \varphi_0(Nx)$ is the '*lifted*' character of $\sigma(x)$ and the representation $\sigma(u)$ of all $u \in N$ is equal to $m \times m$ identity matrix, hence the character of representation $\sigma(u)$ is m. Consequently these data are presented in form of the character table.

Results

The group C_2 consists of the elements $1, a\ (a^2 = 1)$. Then cyclic group C_2 has two irreducible representations

$\chi^{(r)}(a^s) = \exp(2\pi irs/n)$,

$(r, s = 0, 1, \ldots, n-1)$,

here $n = 2$ implies that $r, s = (0, 1)$. The character table of $C_2 \times C_2$ (Table 1) is given by

$$\chi^{(r)}(x) = \exp\left[2\pi i\left(\frac{b_1 r_1}{2} + \frac{b_2 r_2}{2}\right)\right],$$

where $x = a_1^{b_1} a_2^{b_2}$, $r = [r_1, r_2]$ and $b_1, b_2, r_1, r_2 = 0, 1$.

Table1.Character table of $C_2 \times C_2$

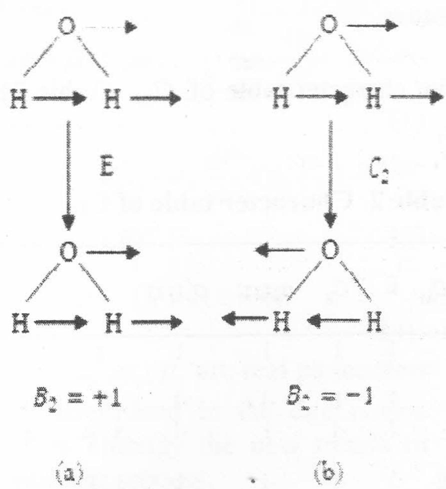
$C_2 \times C_2$	1	a_1	a_2	$a_1 a_2$
$\chi^{(0)}$	1	1	1	1
$\chi^{(1)}$	1	-1	1	-1
$\chi^{(2)}$	1	1	-1	-1
$\chi^{(3)}$	1	-1	-1	1

The character table of any finite direct product of cyclic groups can be obtained similarly. The character table of $C_2 \times C_2$ can be obtained in following way.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & -\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = H_2 \times H_2 \text{ where } H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Since the normal subgroup of alternating group of order 12 (A_4) form a factor group which is isomorphic to C_3, the character table of A_4 can be obtained directly by lifting the character table of C_3. Furthermore, the symmetry element of a molecule such as water has one C_2 axis, two σ_v planes, and E. This set of four symmetry elements (Fig 1) characterizes the point group C_{2v} (cyclic with vertical plane).



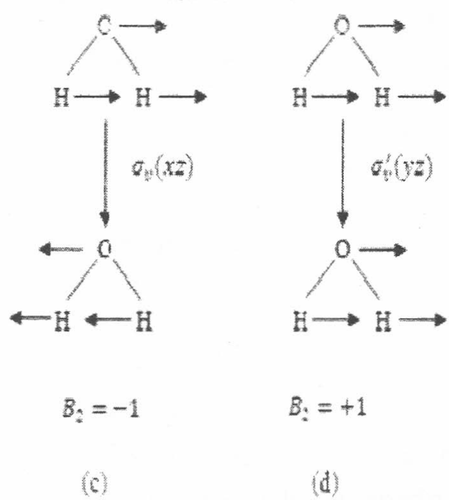


Fig1.Effects of symmetry operations in C_{2v} symmetry; translation along the *y*-axis: (a) identity, E; (b) rotation about the C_2 axis; (c, d) reflection in σ_v planes.

The character table of C_{2v} (Table 2) is,

Table 2. Character table of C_{2v}

C_{2v}	E	C_2	$\sigma_v(xz)$	$\sigma'_v(yz)$		
	1	1	1	1	z	x^2, y^2, z^2
	1	1	-1	-1	R_z	xy
	1	-1	1	-1	x ; R_y	xz
B_2	1	-1	-1	1	y ; R_x	yz

Discussion

As the character table of C_2 is a Hadarmad matrix, the character table of any finite direct product of C_2 into itself can be obtained by Sylvester's construction of Hadamard matrix. The table obtained for the group of rotations of the water molecule is shown to be isomorphic to character table of $C_2 \times C_2$.

References

Gary, L. M. and Donald, A. T. (1991). Inorganic Chemistry 3rd Edition Prentice Hall.

Gordo, D. J. and Martin, W.L. (1993). Representations and Characters of Groups 2nd Edition Cambridge University Press.