MAHLER MEASURE AND THE LEHMER PROBLEM

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Introduction

It is worth noticing that the Mahler Measure appears in different areas of mathematics such as K-theory, number theory, topology and dynamical systems. The logarithm of Mahler Measure is a quantity that occurs naturally as an entropy or growth rate. Mahler Measure also seems to have a close connection to the study of the A-polynomials of arithmetic hyperbolic manifolds.

Let

$$f(x) = \sum_{i=0}^{n} a_i x^i = a_n \prod_{i=1}^{n} (x - \alpha_i),$$

be an n^{th} degree polynomial with roots $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, where $a_i \in \mathbb{N}$ for $i \in \{1, 2, 3, \ldots, n\}$. Then the Mahler Measure of the polynomial $f(x)$ is defined as,

$$M(f) = |a_n| \prod_{i=1}^{n} max\{1, |\alpha_i|\}.$$

It is immediate from the definition of $M(f)$ that $M(f) \geq 1$ and $M(f) = 1$ if f is a product of x or any cyclotomic polynomial.

D. H. Lehmer (1933) posed the following question.

For a given $\varepsilon > 0$, does there exist a noncyclotomic irreducible polynomial with integer coefficients such that $M(f) < 1 + \varepsilon$?

This is known as the Lehmer problem or the Lehmer conjecture. Lehmer found that the Mahler Measure of the polynomial

$$l(x) = 1 - x + x^3 - x^4 + x^5 - x^6 + x^7 - x^9 + x^{10},$$

is $M(l) = 1.17628\ldots$, which remains as the smallest Mahler Measure known - answering the above question that $M(f)$ cannot be made arbitrarily close to 1.

For the family of quadratic polynomials the smallest $M(f)$ other than 1 is the golden ratio $\frac{1+\sqrt{5}}{2}$ which again yields the answer 'no' to the Lehmer problem for this particular class of polynomials.

Here, it has been shown, that the smallest Mahler Measure of an algebraic number α in any quadratic field

$$\mathbb{Q}(\sqrt{m}) = \{a + b\sqrt{m} \mid a, b \in \mathbb{Q}\},$$

where m is a square free integer, is the golden ratio;

i.e. $M(\alpha) = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$.

For some special cases of m, a connection has been drawn between $M(\alpha)$ and $\sqrt{m}$, for any $\alpha \in \mathbb{Q}(\sqrt{m})$ but our goal is to establish a much stronger relationship between the two. Some interesting minimal values of Mahler Measures of higher degree polynomials are also presented.

Given a square free integer m, let $\mathbb{Q}(\sqrt{m})$ denote the quadratic field
$$\mathbb{Q}(\sqrt{m}) = \{ a + b\sqrt{m} \mid a, b \in \mathbb{Q} \}$$
and for any
$$\alpha = a + b\sqrt{m} \in \mathbb{Q}(\sqrt{m}),$$
the Mahler Measure of α is defined as
$$M(\alpha) = M\left(c\left(x^2 - 2ax + a^2 - b^2 m\right)\right)$$
where c is the smallest integer value to clear off the denominators.
It has been shown (Schinzel, 1973) that if α lies in a kroneckerian field (a totally real number field, or a totally quadratic extension of such a field) and $|\alpha| \neq 1$, then
$$M(\alpha) \geq \frac{1+\sqrt{5}}{2};$$
this is a much broader class than the class of quadratic fields.
Furthermore it has been shown (Ishak et al., 2010) that an improved bound can be found for this class where a non zero algebraic number α that lies in an abelian extension of the rationals and is not a root of unity must satisfy the condition
$$\log M(\alpha) > (0.155097)d,$$
where d is the degree of the minimal polynomial of α. Evidently this emphasis is on a broader class of polynomials than that on the above (Schinzel, 1973).

Results

From the above we know that
$$M\left(\frac{1+\sqrt{5}}{2}\right) = M(x^2 - x - 1) = \frac{1+\sqrt{5}}{2}$$
is obtained in $\mathbb{Q}(\sqrt{5})$. We would like to see how $M(\alpha)$ changes with $\sqrt{m}$, where $\alpha \in \mathbb{Q}(\sqrt{m})$. Then the question would be how close is it to $\sqrt{m}$?
Our goal is to find possible smallest Mahler Measure in higher degrees. Some interesting smaller Mahler Measures can be found out for 3^{rd}, 8^{th} and 10^{th} degree polynomials.

$f_1(x) = x^3 - x + 1$ yields the value $M(f_1) = 1.3247...$

$f_2(x) = x^8 - x^5 - x^4 - x^3 + 1$ yields the value $M(f_2) = 1.2806...$

$f_3(x) = x^{10} - x^6 + x^5 - x^4 + 1$ yields the value $M(f_3) = 1.2164...$

A polynomial $f(x)$ is said to be reciprocal if $f(x) = x^n f\left(\frac{1}{x}\right)$ where n is the degree of the polynomial. It is known that (Smyth, 1971) if $f(x)$ is an irreducible polynomial and
$$M(f) < 1.1762802,$$
then $f(x)$ is reciprocal. In this direction we try to compute the reciprocal polynomials $f(x)$ having smaller Mahler Measures such that, for given constants $M_1 > 0$ and $n_1 \in \mathbb{N}$ the degree of $f(x) \leq n_1$ and $M(f) \leq M_1$

References

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